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# ENVELOPING ALGEBRAS WITH JUST INFINITE GELFAND-KIRILLOV DIMENSION

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**ABSTRACT.** Let  $\mathfrak{g}$  be the Witt algebra or the positive Witt algebra. It is well known that the enveloping algebra  $U(\mathfrak{g})$  has intermediate growth and thus infinite Gelfand-Kirillov (GK-) dimension. We prove that the GK-dimension of  $U(\mathfrak{g})$  is *just infinite* in the sense that any proper quotient of  $U(\mathfrak{g})$  has polynomial growth. This proves a conjecture of Petukhov and the second named author for the positive Witt algebra. We also establish the corresponding results for quotients of the symmetric algebra  $S(\mathfrak{g})$  by proper Poisson ideals.

In fact, we prove more generally that any central quotient of the universal enveloping algebra of the Virasoro algebra has just infinite GK-dimension. We give several applications. In particular, we easily compute the annihilators of Verma modules over the Virasoro algebra.

## 1. INTRODUCTION

Let  $\mathbb{K}$  be a field of characteristic zero, and let  $W$  be the *Witt algebra*, which has  $\mathbb{K}$ -basis

$$\{e_n : n \in \mathbb{Z}\},$$

with the Lie bracket

$$[e_i, e_j] = (j - i)e_{i+j}.$$

We let  $W_+$ , the *positive Witt algebra*, be the Lie subalgebra of  $W$  spanned by  $\{e_n : n \geq 1\}$ .

The Witt algebra is a central quotient of the *Virasoro algebra*,  $Vir$ , which has  $\mathbb{K}$ -basis

$$\{e_n : n \in \mathbb{Z}\} \cup \{c\},$$

and Lie bracket

$$[e_i, e_j] = (j - i)e_{i+j} + \frac{i^3 - i}{12}\delta_{i+j,0}c, \quad c \text{ central}.$$

The Virasoro algebra and its representations are ubiquitous in conformal field theory.

These algebras were testing examples for the fundamental and important question of whether there is an infinite-dimensional Lie algebra with a (left and right) noetherian enveloping algebra. This question has been asked by many people, including Ralph Amayo and Ian Stewart [AS74, Question 27, p. 396], Ken Brown [Bro07, Question B], Jacques Dixmier, and Victor Latyshev, and conjecturally has a negative answer; see [SW14, Conjecture 0.1]. Recently it was shown by the second named author and Chelsea Walton [SW14] that the conjecture holds for the Lie algebras above: that is,  $U(W_+)$ ,  $U(W)$ , and  $U(Vir)$  are not left or right Noetherian. The question is still unsolved in full generality.

However, the two-sided ideal structure of these enveloping algebras is extremely sparse, and it seems possible that they satisfy the ascending chain condition for two-sided ideals, a

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property sometimes known as being *weakly Noetherian*. Further, two-sided ideals of  $W$  and  $W_+$  are known to be large, and Petukhov and the second named author have conjectured:

**Conjecture 1.1.** ([PS17, Conjecture 1.2]) *The universal enveloping algebra  $U(W_+)$  has just infinite GK-dimension; that is, if  $I$  is a non-zero ideal of  $U(W_+)$ , then  $U(W_+)/I$  has polynomial growth.*

(In this paper, we use *polynomial growth* as a synonym for finite GK-dimension.)

This conjecture was proved in [PS17] in the particular case that the ideal  $I$  is generated by quadratic expressions in the  $e_i$ . In this paper we establish the conjecture in full, and generalise our arguments to prove that  $U(W)$  and indeed any central factor of  $U(\text{Vir})$  has just infinite GK-dimension in the sense above. Our main result is:

**Theorem 1.2.** (See Theorems 3.4, 4.1 and 5.3.) *The algebras  $U(W_+)$ ,  $U(W)$ , and  $U(\text{Vir})/(c - \kappa)$ , for any  $\kappa \in \mathbb{K}$ , have just infinite GK-dimension. In particular, Conjecture 1.1 holds.*

We note that these algebras are all finitely generated and as many authors have noticed [Smi76, Ufn78, KKM83] have intermediate growth (and thus infinite GK-dimension). The natural set of normal words is monomials of the form  $e_{i_1}, \dots, e_{i_k}$ ,  $i_1 \leq i_2 \leq \dots \leq i_k$ . Here the number of monomials with  $i_1 + i_2 + \dots + i_k = n$  is in fact the number of partitions of  $n$ , which was shown by Hardy to be bounded by  $e^{c\sqrt{n}}$ , an estimate later dramatically improved by Ramanujan.

We also consider the induced Poisson structures on the symmetric algebras  $S(W_+)$ ,  $S(W)$ , and  $S(\text{Vir})$ , and prove:

**Theorem 1.3.** (See Theorems 2.1 and 5.6.) *Let  $I$  be a proper Poisson ideal of  $S(W_+)$ . Then  $S(W)/I$  has polynomial growth. Similar statements hold for  $S(W)$  and for  $S(\text{Vir})/(c - \kappa)$ , for any  $\kappa \in \mathbb{K}$ .*

The Lie algebra  $\text{Vir}$  has a triangular decomposition, and so the classical notion of a *Verma module* makes sense. As an application of Theorem 1.2, we easily compute annihilators of Verma modules for  $\text{Vir}$ :

**Theorem 1.4.** (see Corollary 6.2) *Let  $M$  be a Verma module over  $\text{Vir}$  with central charge  $\kappa$ . Then the annihilator of  $M$  in  $U(\text{Vir})$  is the ideal  $(c - \kappa)$ . As a result, for any  $\kappa \in \mathbb{K}$  the algebra  $U(\text{Vir})/(c - \kappa)$  is primitive.*

Theorem 1.4 is an unpublished result of Wallach [WS13].

We also obtain:

**Proposition 1.5.** (See Proposition 6.4.) *The algebras  $U(W)$ ,  $U(W_+)$ , and  $U(\text{Vir})$  all satisfy the ascending chain condition for completely prime ideals.*

**Proposition 1.6.** (See Proposition 6.5.) *Let  $U(\mathcal{W})$  be either  $U(W_+)$ ,  $U(\text{Vir})$ , or  $U(\text{Vir})/(c - \kappa)$  for some  $\kappa \in \mathbb{K}$ . Then any epimorphism of  $U(\mathcal{W})$  is an isomorphism.*

This answers a question of Rowen and Small [RS17, Section 4].

The Witt algebra  $W$  is a simple, graded, Lie algebra of polynomial growth. Such algebras were famously classified by Mathieu [Mat92]. It is interesting to ask which of these Lie algebras have enveloping algebras with just infinite GK-dimension. This is the subject of ongoing research.

**Methods:** As notation, we will write the symmetric algebra  $S(W_+) = \mathbb{K}[x_1, x_2, \dots]$ , where  $x_i$  corresponds to  $e_i \in W_+$ . Likewise,

$$S(W) = \mathbb{K}[\dots, x_{-1}, x_0, x_1, \dots] \quad \text{and} \quad S(\text{Vir}) = \mathbb{K}[\dots, x_{-1}, x_0, x_1, \dots, c].$$

The main idea in the proofs of both Theorems 1.2 and 1.3 is to show that if  $g$  is a nonzero element of  $S(W_+)$  or  $S(W)$ , then for ‘almost all’ monomials  $m$  in the  $x_i$  (more precisely, for long enough monomials on big enough letters), the Poisson ideal generated by  $g$  contains an element with leading term  $m$ . (See Lemma 2.2.) In the same way, we show that if  $g$  is a nonzero element of  $U(W_+)$  or  $U(W)$ , then for almost all monomials  $m$  in the variables  $e_i$ , the two-sided ideal generated by  $g$  contains an element with leading term  $m$ . (See Lemmata 3.1 and 4.4.)

We summarise the argument for  $S(W_+)$ . Let  $I$  be the Poisson ideal generated by  $0 \neq g \in S(W_+)$ . We introduce the following ordering on monomials in the  $x_i$ . Denote by  $\deg(m) = i_1 + \dots + i_k$  the *degree* of a monomial  $m = x_{i_1} \dots x_{i_k}$ , and let the *length* of  $m$  be  $|m| = |x_{i_1} \dots x_{i_k}| = k$ .

Then the ordering on the set of commutative monomials in the  $x_i$  is defined as follows. For two monomials  $m_1$  and  $m_2$ , we write  $m_1 < m_2$  if

- $|m_1| < |m_2|$  or
- $|m_1| = |m_2|$  and  $\deg(m_1) < \deg(m_2)$  or
- $|m_1| = |m_2|$ ,  $\deg(m_1) = \deg(m_2)$  and  $m_1$  is less than  $m_2$  with respect to the left-to-right lexicographical order when both  $m_1$  and  $m_2$  are written in increasing order:  
 $m_1 = x_{i_1} \dots x_{i_k}$ ,  $m_2 = x_{j_1} \dots x_{j_k}$  with  $i_1 \leq i_2 \leq \dots \leq i_k$ ,  $j_1 \leq j_2 \leq \dots \leq j_k$ .

We show in Lemma 2.2 that all sufficiently long monomials on sufficiently ‘big’ letters can be written, modulo  $I$ , as a sum of smaller monomials. Here a letter is called *big* if it is bigger than  $n = \{\max(2i + 1) \mid x_i \text{ occurs in } g\}$ . By this means we are able to introduce a ‘normal form’ for monomials from  $S(W_+)$ . Namely, any element of  $S(W_+)$  can be written, modulo  $I$ , as a linear combination of monomials of the form  $m = uv$ , where  $u$  is a monomial on the finite set of letters  $x_1, \dots, x_{n-1}$ , and  $v$  is a monomial of restricted length on the set of big letters. A similar normal form works for  $U(W_+)$ .

In the case of  $W$  (more generally, central quotients of  $\text{Vir}$ ) the normal form is slightly different. Let  $0 \neq J \triangleleft U(W)$ . Any element of  $U(W)$  can be written, modulo  $J$ , as a linear combination of monomials  $m = u_1 v u_2$ , where  $v$  is a monomial on the finite set of letters  $e_{1-n}, \dots, e_{n-1}$ ,  $u_1$  is a monomial of bounded length on letters smaller than  $e_{-n}$ , and  $u_2$  is a monomial of bounded length on letters larger than  $e_n$ .

Counting the growth of these normal monomials gives us a polynomial estimate which bounds the growth of the quotient algebra. In the case of the full Witt algebra the growth counting is somewhat more involved, since the usual degree function  $\deg(u) = i_1 + \dots + i_k$  will not supply us a finite filtration on  $U(W)/J$ . See Section 4.2 for the details of how this issue is resolved.

**Notation:** Throughout we fix the following notation. We denote the set of non-negative integers by  $\mathbb{N}$ . If  $R$  is a ring and  $g \in R$ , the two-sided ideal generated by  $g$  is denoted  $(g)$ . If  $R$  is a Poisson algebra, the Poisson ideal generated by  $g$  is denoted  $\{(g)\}$ .

Let  $S(\mathcal{W})$  denote either  $S(W_+)$ ,  $S(\text{Vir})$ , or  $S(\text{Vir})/(c - \kappa)$  for some  $\kappa \in \mathbb{K}$ . Likewise, let  $U(\mathcal{W})$  be either  $U(W_+)$ ,  $U(\text{Vir})$ , or  $U(\text{Vir})/(c - \kappa)$ . Our convention is that  $e_1, e_2, \dots$  denote elements of  $U(\mathcal{W})$ , and that  $x_1, x_2, \dots$  are the corresponding elements of  $S(\mathcal{W})$ . Monomials

$x_{i_1}x_{i_2}\dots x_{i_k} \in S(\mathcal{W})$  are usually written in increasing order (if not specified otherwise), that is  $i_1 \leq i_2 \leq \dots \leq i_k$ . The same shape  $e_{i_1}e_{i_2}\dots e_{i_k}$ , where  $i_1 \leq i_2 \leq \dots \leq i_k$ , of a monomial is considered normal in  $U(\mathcal{W})$ , and such a monomial is called *standard*.

## 2. THE SYMMETRIC ALGEBRA OF THE POSITIVE WITT ALGEBRA

Our main goal in this section is to prove the following theorem, which gives the first statement of Theorem 1.3:

**Theorem 2.1.** *Let  $I$  be a nonzero Poisson ideal of  $S(W_+)$ . Then  $A = S(W_+)/I$  has polynomial growth.*

Let  $X = \{x_1, x_2, \dots\}$ , so  $S(W_+) = \mathbb{K}[X]$ , with Poisson bracket induced by defining  $\{x_i, x_j\} = (j-i)x_{i+j}$ . We equip  $S(W_+)$  with the degree grading given by setting  $\deg(x_j) = j$ . Thus the *degree* of a monomial  $m = x_{i_1}\dots x_{i_k}$  is  $\deg(m) = i_1 + \dots + i_k$ . Of course, we have the natural grading by the *length* of monomials as well, which we write as  $|x_{i_1}\dots x_{i_k}| = k$ .

The key technique in the proof is the following *reduction formula* for elements of  $A = S(W_+)/I$  in the case that  $I = \{(g)\}$  is the Poisson ideal generated by a single, nonzero, degree-homogeneous polynomial  $g$ .

**Lemma 2.2.** *Let  $I = \{(g)\}$  be the Poisson ideal in  $\mathbb{K}[X] = S(W_+)$  generated by a nonzero degree-homogeneous polynomial  $g \in \mathbb{K}[X]$ . There exist positive integers  $k$  and  $n$  such that every monomial  $m = x_{j_1}\dots x_{j_k}$  such that  $j_\ell \geq n$  for  $1 \leq \ell \leq k$  satisfies*

$$(2.3) \quad m = h + \sum c_s m_s,$$

where  $h \in I$  is degree-homogeneous with  $\deg(h) = \deg(m)$ , the sum is finite,  $c_s \in \mathbb{K}$ , and the  $m_s$  are monomials of degree  $\deg(m)$  such that for each  $s$ , either  $|m_s| < k$  or  $|m_s| = k$  and  $i < n$  for at least one of the letters  $x_i$  featuring in  $m_s$ .

To prove Lemma 2.2, we introduce the following two orderings on the set  $[X]$  of commutative monomials in  $X$ . For two monomials  $m_1$  and  $m_2$ , we write  $m_1 < m_2$  if

- $|m_1| < |m_2|$  or
- $|m_1| = |m_2|$  and  $\deg(m_1) < \deg(m_2)$  or
- $|m_1| = |m_2|$ ,  $\deg(m_1) = \deg(m_2)$  and  $m_1$  is less than  $m_2$  with respect to the left-to-right lexicographic order when both  $m_1$  and  $m_2$  are written in *increasing* order:  
 $m_1 = x_{i_1}\dots x_{i_k}$ ,  $m_2 = x_{j_1}\dots x_{j_k}$  with  $i_1 \leq i_2 \leq \dots \leq i_k$ ,  $j_1 \leq j_2 \leq \dots \leq j_k$ .

Similarly, we write  $m_1 \prec m_2$  if

- $|m_1| < |m_2|$  or
- $|m_1| = |m_2|$  and  $\deg(m_1) < \deg(m_2)$  or
- $|m_1| = |m_2|$ ,  $\deg(m_1) = \deg(m_2)$  and  $m_1$  is less than  $m_2$  with respect to the left-to-right lexicographical order when both  $m_1$  and  $m_2$  are written in *decreasing* order:  
 $m_1 = x_{i_1}\dots x_{i_k}$ ,  $m_2 = x_{j_1}\dots x_{j_k}$  with  $i_k \leq i_{k-1} \leq \dots \leq i_1$ ,  $j_k \leq j_{k-1} \leq \dots \leq j_1$ .  
(Equivalently, we may write  $m_1$  and  $m_2$  in increasing order and compare them with the right-to-left lexicographic order.)

Note, that in the ordering  $<$  we compare the smallest letters first, and in the ordering  $\prec$  we compare the biggest letters first. It is easy to see that both orderings are well-orderings on  $[X]$  compatible with multiplication.

*Proof of Lemma 2.2.* Our ordering  $<$  satisfies the descending chain condition, and thus Lemma 2.2 can easily be obtained by repeated application of the following sublemma:

**Sublemma:** There exist positive integers  $k$  and  $n$  such that every monomial  $m \in [X]$  of length  $k$  with  $m \geq x_n^k$  satisfies

$$m = h + \sum c_s m_s,$$

where  $h \in I$  is degree-homogeneous with  $\deg(h) = \deg(m)$ , the sum is finite,  $c_s \in \mathbb{K}$ , and the  $m_s$  are monomials of degree  $\deg(m)$  such that  $m_s < m$  for each  $s$ .

To prove the sublemma, let  $\bar{g}$  be the leading monomial of  $g$  with respect to  $\prec$ , and let  $k = |\bar{g}| = |g|$ . Without loss of generality we can assume that  $\bar{g}$  features in  $g$  with coefficient 1. We write  $\bar{g}$  in an increasing way:  $\bar{g} = x_{i_1} \dots x_{i_k}$  with  $i_1 \leq i_2 \leq \dots \leq i_k$ . Pick any positive integer  $n$  such that  $n \geq 2i_k + 1$ . We shall show that these  $n$  and  $k$  satisfy the required conditions.

Let  $m \in [X]$  of length  $k$  be such that  $m \geq x_n^k$ . Then  $m = x_{j_1} \dots x_{j_k}$  with  $n \leq j_1 \leq \dots \leq j_k$ . If  $a \in \mathbb{Z}_{\geq 1}$ , let  $d_a : \mathbb{K}[X] \rightarrow \mathbb{K}[X]$  be the derivation defined by  $d_a(u) = \{u, x_a\}$ , extending via the Leibniz rule. Note that  $d_a$  is a graded derivation: if applied to a degree-homogeneous polynomial  $f$ , then  $d_a(f)$  is degree-homogeneous of degree  $\deg(d_a(f)) = \deg(f) + a$ .

Consider

$$h = d_{j_1 - i_k} d_{j_2 - i_{k-1}} \dots d_{j_k - i_1}(g).$$

Since  $g \in I$  and  $I$  is a Poisson ideal,  $h \in I$ . Further, as the  $d_a$  are graded and  $g$  is degree-homogenous, so is  $h$ . The proof will be complete if we verify that

$$h = cm + \sum c_s m_s,$$

where  $c \neq 0$ ,  $c_s \in \mathbb{K}$ , and the  $m_s$  are monomials such that  $m_s < m$  for each  $s$ .

Let us apply the sequence of derivations  $d_{j_1 - i_k} d_{j_2 - i_{k-1}} \dots d_{j_k - i_1}$  to a monomial  $u$  occurring in  $g$ . By the Leibniz rule we get a sum of monomials with coefficients obtained by prescribing which of the derivations acts on each letter of  $u$ .

Note (assuming that  $d_a(u) \neq 0$ ) that  $d_a(u)$  has the same length as  $u$ . Thus monomials in  $h$  obtained from monomials in  $g$  of length  $< k$  are themselves of length  $< k$  and therefore are smaller than  $m$  with respect to  $<$ .

Suppose now that  $u$  has length  $k$ . Then there are two options: either different differentials act on letters in different places in the monomial  $u$  or this is not the case. We call the first of these ways *permutational* and the second *non-permutational*. Monomials in  $h$  obtained from  $u$  in a non-permutational way will have at least one letter unchanged and therefore will have at least one letter  $x_i$  with  $i \leq i_k < n$ . Hence such monomials of  $h$  are again smaller than  $m$  with respect to  $<$ .

It remains to consider monomials of  $h$  obtained from monomials in  $g$  of length  $k$  in a permutational way. For each monomial  $u = x_{p_1} \dots x_{p_k}$  with  $p_1 \leq \dots \leq p_k$  occurring in  $g$  and each permutation  $\sigma \in \mathfrak{S}_k$ , we obtain a monomial  $w$  of  $h$  given by

$$w = x_{j_1 - i_k + p_{\sigma(k)}} \dots x_{j_k - i_1 + p_{\sigma(1)}},$$

occurring with coefficient

$$\prod_{\ell=1}^k (j_\ell - i_{k-\ell+1} - p_{\sigma(k-\ell+1)}).$$

Since  $u \preceq \bar{g}$ , thus  $p_{\sigma(k)} \leq p_k \leq i_k$ . Hence  $j_1 - i_k + p_{\sigma(k)} \leq j_1$  and the equality  $j_1 - i_k + p_{\sigma(k)} = j_1$  holds if and only if  $p_{\sigma(k)} = p_k = i_k$ . If  $j_1 - i_k + p_{\sigma(k)} < j_1$ , then  $w < m$  and we are done. If  $j_1 - i_k + p_{\sigma(k)} = j_1$ , we have  $p_{\sigma(k)} = p_k = i_k$ . Since  $u \preceq \bar{g}$  and  $p_{\sigma(k)} = p_k = i_k$ , we have  $p_{\sigma(k-1)} \leq p_{k-1} \leq i_{k-1}$ . Hence  $j_2 - i_{k-1} + p_{\sigma(k-1)} \leq j_2$  and the equality  $j_2 - i_{k-1} + p_{\sigma(k-1)} = j_2$  holds if and only if  $p_{\sigma(k-1)} = p_{k-1} = i_{k-1}$ . Repeating the procedure, we see that  $w \leq m$  and that  $w = m$  only if  $u = \bar{g}$  and the permutation  $\sigma$  satisfies  $i_{\sigma(s)} = i_s$  for  $1 \leq s \leq k$ .

Now, for each such  $\sigma$ , since  $n \geq 2i_k + 1$ , we claim that the coefficient of the monomial  $m$  is a positive integer. Indeed, the coefficient is a product of factors of the form  $j_\ell - i_{k-\ell+1} - p_{\sigma(k-\ell+1)}$ , which are positive since  $i_{k-\ell+1} + p_{\sigma(k-\ell+1)} \leq 2i_k < n \leq j_\ell$ . Thus the coefficient with which  $m$  occurs in  $h$  is nonzero. The other monomials in  $h$  are  $< m$ . This completes the proof of the sublemma and thus of the lemma.  $\square$

Note that we used in this proof only that  $I$  is a module over  $\mathbb{K}$  defined by the bracket multiplication (in other words a submodule of  $S(W_+)$  under the adjoint action of  $W_+$ ).

We now prove Theorem 2.1. The key point of the proof is that, thanks to Lemma 2.2,  $A = S(W_+)/I$  is spanned by the set of monomials  $m$  in  $[X]$  which admit a factorisation  $m = m_1 m_2$ , where  $m_1$  is a monomial in  $x_1, x_2, \dots, x_{n-1}$  and  $|m_2| < k$ . Thus to estimate the growth of  $A$  it suffices to count such monomials.

*Proof of Theorem 2.1.* As  $S(W_+)$  is finitely graded by degree, it is standard (see [KL00, Proposition 6.6]) that it suffices to show that  $S(W_+)/I$  has polynomial growth if  $I$  is a nonzero degree-graded Poisson ideal, and it clearly suffices to consider the case that  $I$  is the Poisson ideal generated by a single nonzero degree-homogenous element  $g$ . Let  $k$  and  $n$  be the numbers produced by applying Lemma 2.2 to  $g$ , and let  $S$  be the set of all monomials  $m$  in  $[X]$  which admit a factorisation  $m = m_1 m_2$ , where  $m_1$  is a monomial in  $x_1, x_2, \dots, x_{n-1}$  and  $|m_2| < k$ .

By Lemma 2.2, each monomial  $m \in [X] \setminus S$  can be written, modulo  $I$ , as a linear combination of monomials of degree  $\deg(m)$ , each of which either has length strictly less than  $|m|$  or has length  $|m|$  and features strictly fewer  $x_i$  with  $i \geq n$ . Applying this observation repeatedly, we see that every monomial  $m \in [X] \setminus S$  can be written, modulo  $I$ , as a linear combination of monomials from  $S$  of the same degree as  $m$ . We will call such presentation a *normal form* of  $m$ . Hence the image of  $S$  in  $A = \mathbb{K}[X]/I$  spans  $A$ , and for fixed  $N \in \mathbb{N}$ , the number of monomials in  $S$  of degree not exceeding  $N$  provides an upper bound for  $\dim \{u \in A \mid \deg u \leq N\}$ . As  $A$  is finitely  $\mathbb{N}$ -graded by degree, it is standard that the growth of this dimension bounds  $\text{GKdim } A$ .

It remains to estimate the number  $p(N)$  of elements of  $S$  of degree at most  $N$ . Clearly  $p(N) \leq q(N)r(N)$ , where  $q(N)$  is the number of monomials in  $x_1, \dots, x_{n-1}$  of degree at most  $N$  and  $r(N)$  is the number of monomials in  $x_1, x_2, \dots$  of degree at most  $N$  and length at most  $k-1$ . First,  $q(N)$  does not exceed the number of monomials in  $x_1, \dots, x_{n-1}$  of length at most  $N$ , which is  $\binom{N+n}{n}$ . Thus there is a positive constant  $c$  so that  $q(N) \leq cN^n$  for all  $N$ . On the other hand, in a degree at most  $N$  monomial of length at most  $k-1$  in  $x_n, x_{n+1}, \dots$  there are no more than  $N$  options for each letter and therefore  $r(N) \leq N^{k-1}$  for all  $N$ . Hence  $p(N) \leq cN^{k+n-1}$ .

Hence  $\text{GKdim}(A) \leq k + n - 1$ , and  $A$  has polynomial growth, as required.  $\square$

We note that Theorem 2.1 is also proved in [PS17] (see Corollary 2.13), with a much less constructive proof.

To end the section, we fix a positive integer  $k$  and consider the symmetric power  $S^k(W_+)$ . Let  $g \in S^k(W_+)$  be a nonzero degree-homogeneous element, and let  $I$  be the  $W_+$ -submodule of  $S^k(W_+)$  generated by  $g$ . As noted after the proof of Lemma 2.2, the reduction formula in Lemma 2.2 still applies to  $S^k(W_+)/I$ , and the argument in the proof of Theorem 2.1 now gives that  $\text{GKdim } S^k(W_+)/I \leq k - 1$ . As  $S^k(W_+)$  clearly has GK-dimension  $k$ , we obtain:

**Proposition 2.4.** *As a  $W_+$ -module,  $S^k(W_+)$  is GK  $k$ -critical.*  $\square$

For  $k = 2$ , this was shown in [PS17, Corollary 4.15].

### 3. THE UNIVERSAL ENVELOPING ALGEBRA OF THE POSITIVE WITT ALGEBRA

That  $U(W_+)$  has just infinite GK-dimension follows from Theorem 2.1 using the Poisson GK-dimension defined in [PS17] and [PS17, Theorem 3.19]. However, a direct proof, which we give here, is also straightforward; the techniques of Section 2 apply also to  $U(W_+)$ .

We begin by giving a noncommutative version of the reduction formula of Lemma 2.2. Our result is more general than needed here, for later use when considering quotients of  $U(W)$ .

By the Poincaré-Birkhoff-Witt theorem,  $U(W)$  has a basis of monomials  $e_{i_1}e_{i_2}\dots e_{i_k}$  with  $i_1 \leq i_2 \leq \dots \leq i_k$ . We call such monomials *standard*.

**Lemma 3.1.** *Let  $0 \neq g \in U(W)$ , and let  $I = U(W_+)gU(W_+)$  be the  $U(W_+)$ -sub-bimodule of  $U(W)$  generated by  $g$ . Then there exist positive integers  $k$  and  $n$  and an integer  $\ell$  so that every standard monomial  $m = e_{j_1}\dots e_{j_k}$  with  $n \leq j_1 \leq \dots \leq j_k$  satisfies*

$$(3.2) \quad m = h + \sum c_t m_t,$$

where  $h \in I$ , the sum is finite,  $c_t \in \mathbb{K}$ , and the  $m_t$  are standard monomials so that for each  $t$ , we have  $i \geq \ell$  for all letters  $e_i$  featuring in  $m_t$ , and either  $|m_t| < k$  or  $|m_t| = k$  and  $i < n$  for at least one of the letters featuring in  $m_t$ . Further, if  $h$  is degree-homogeneous, then  $\deg(m) = \deg(h) = \deg(m_t)$  for all  $t$ .

*Proof.* Let  $k = |g|$ . Writing  $g$  as a sum of standard monomials, let  $e_\ell$  be the smallest letter occurring in  $g$  and define  $n'$  so that  $e_{n'}$  is the largest letter in  $g$ . Let  $n = 2|n'| + 1$ .

There are well-defined monomial orderings  $<$  and  $\prec$  on standard monomials in  $U(W)$ , defined just as the corresponding orderings on commutative monomials  $x_{i_1}\dots x_{i_k}$  in the previous section. Note that  $<$  does not satisfy the descending chain condition because  $W$  has no least element, but the induced order on standard monomials in letters  $\geq \ell$  does satisfy d.c.c. Thus, as in the proof of Lemma 2.2, it suffices to show that we can rewrite  $m$ , modulo  $I$ , as a linear combination of standard monomials in letters  $\geq \ell$ , each of which are  $< m$ .

For any  $a \in \mathbb{Z}$ , let  $\partial_a = [\_, e_a]$  as a linear operator from  $U(W) \rightarrow U(W)$ . Recall that length defines a filtration on  $U(W)$  whose associated graded ring is  $S(W)$ ; for  $f \in U(W)$  let  $\text{gr}(f)$  be the element of  $S(W)$  associated to  $f$ , so  $x_i = \text{gr}(e_i)$ . For any  $p \in U(W)$  and any  $a \in \mathbb{Z}$ , we have

$$(3.3) \quad \text{gr } \partial_a(p) = d_a(\text{gr}(p)) \quad \text{if } d_a(\text{gr}(p)) \neq 0,$$

where  $d_a = \{\_, x_a\}$  as in the proof of Lemma 2.2.

Let  $\bar{g}$  be the  $\prec$ -leading standard monomial in  $g$  and write  $\bar{g} = e_{i_1}\dots e_{i_k}$  with  $i_1 \leq \dots \leq i_k$ . Let

$$h = \partial_{j_1-i_k} \partial_{j_2-i_{k-1}} \dots \partial_{j_k-i_1}(g).$$



Since for any letter  $x_p$  featuring in  $\text{gr}(g)$ , and for any  $d_a$  which is applied to  $x_p$ , by our choice of  $n$  we have  $a > p$ , as in the proof of Lemma 2.2. Thus, just as in that proof, and using (3.3),

$$\text{gr}(h) = d_{j_1-i_k} d_{j_2-i_{k-1}} \cdots d_{j_k-i_1}(\text{gr}(g)) = c \text{gr}(m) + \sum c_t \text{gr}(m'_t),$$

where  $c \neq 0$ , the sum is finite,  $c_t \in \mathbb{K}$ , and the  $m'_t$  are standard monomials so that  $m'_t < m$  for all  $t$ . Thus the length of  $h - m - \sum c_t m'_t$  is strictly smaller than  $k$ , so  $h - m$  is a linear combination of standard monomials which are all strictly  $< m$ .

Since  $a > 0$  for all  $\partial_a$  we have applied, only letters  $\geq \ell$  occur in  $h$ . Finally, as the  $\partial_a$  are graded linear maps, if  $g$  is degree-homogenous so is  $h$ .  $\square$

**Theorem 3.4.** *Let  $I$  be a nonzero two-sided ideal in  $U(W_+)$ . Then  $A = U(W_+)/I$  has polynomial growth.*

*Proof.* All essential points of the proof occur the proof of Theorem 2.1.

As before, since  $U(W_+)$  is finitely graded by degree, we may assume that  $I = (g)$  is the ideal generated by a single nonzero degree-homogenous element  $g$ . Let  $k = |g|$ .

Let  $S$  be the set of standard monomials  $m$  which admit a factorization  $m = m_1 m_2$ , where  $m_1$  is a standard monomial in  $e_1, \dots, e_{n-1}$  and  $|m_2| < k$ . It follows from the reduction formula in Lemma 3.1 that  $U(W_+)/I$  is spanned by the image of  $S$ . The same counting argument as in the proof of Theorem 2.1 shows that  $U(W_+)/I$  has polynomial growth.  $\square$

Theorem 3.4 gives the first part of Theorem 1.2, dealing with  $U(W_+)$ .

#### 4. THE ENVELOPING ALGEBRA OF THE FULL WITT ALGEBRA

In this section, we consider the enveloping algebra of the full Witt algebra, and show that it has just infinite GK-dimension. It clearly suffices to show:

**Theorem 4.1.** *Let  $I = (g)$  be a two-sided ideal in  $U(W)$  generated by one nonzero element  $g \in U(W)$ . Then  $A = U(W)/I$  has polynomial growth.*

Throughout the section, we fix the meanings of  $g$ ,  $I$ , and  $A$  as in the statement of Theorem 4.1. Let  $\pi : U(W) \rightarrow A$  be the natural map.

Now,  $U(W)$  is finitely generated, say by  $\{e_{-2}, e_{-1}, e_1, e_2\}$ , and thus so is  $A$ . Since the growth of  $A$  is controlled by the growth of any finite filtration, we are free to choose one that is convenient, but it will be a little bit more complicated this time to choose the right one. The problem is that unlike the situation for  $W_+$ , the usual degree function  $\deg(e_i) = i$  does not give us a finite grading on  $U(W)$ ; note that in the proofs of Theorems 2.1 and 3.4, the finiteness of the degree grading played a crucial role. Moreover, although of course there are many finite filtrations on  $U(W)$ , it is not necessarily clear how to choose one which induces a filtration on the quotient with polynomial growth.

Thus we will need to find an appropriate degree function which will give us a well-controlled finite filtration on  $A$ . We will see that a degree function of the form  $\delta_C$ , defined by  $\delta_C(e_{i_1} \cdots e_{i_k}) = |i_1| + \cdots + |i_k| + C$ , where  $C$  is a constant, does the job.

**4.1. A spanning set for  $A$ .** Our first step is to construct a set of standard monomials in  $U(W)$  whose images span  $A$ .

Symmetrically to Lemma 3.1, we have:

**Lemma 4.2.** *There exist positive integers  $k$  and  $n$  and an integer  $\ell$  such that every standard monomial  $m = e_{j_1} \dots e_{j_k}$  with  $j_1 \leq \dots \leq j_k \leq -n$  satisfies*

$$m = h + \sum c_t m_t,$$

where  $h \in I$ , the sum is finite,  $c_t \in \mathbb{K}$ , and the  $m_t$  are standard monomials so that for each  $t$ , we have  $i \leq \ell$  for all letters  $e_i$  featuring in  $m_t$ , and either  $|m_t| < k$  or  $|m_t| = k$  and  $i > -n$  for at least one of the letters featuring in  $m_t$ .

*Proof.* Consider the automorphism  $\Phi$  of  $U(W)$  defined by  $\Phi(x_i) = -x_{-i}$  for  $i \in \mathbb{Z}$ . If we apply Lemma 3.1 to the ideal  $\Phi(I)$  and the monomial  $\pm\Phi(m)$ , we obtain that  $\Phi(m) = h + \sum c_t m_t$ , where  $h \in I$ , the sum is finite,  $c_t \in \mathbb{K}$ , and the  $m_t$  are standard monomials so that for each  $t$ , we have  $i \geq \ell$  for all letters  $e_i$  featuring in  $m_t$ , and either  $|m_t| < k$  or  $|m_t| = k$  and  $i < n$  for at least one of the letters featuring in  $m_t$ . Applying  $\Phi$  to both sides once again, we arrive at the result.  $\square$

Lemmata 3.1 and 4.2 allow us to construct our spanning set, which we define here.

**Definition 4.3.** For positive integers  $k, n$ , let  $NS(k, n)$  be the set of standard monomials  $m$  which admit a factorisation  $m = aub$ , where  $a$  is a standard monomial of length  $< k$  in  $e_{-n}$  and smaller letters,  $u$  is a standard monomial in  $e_{1-n}, \dots, e_0, e_1, e_2, \dots, e_{n-1}$ , and  $b$  is a standard monomial of length  $< k$  in  $e_n$  and bigger letters.

**Lemma 4.4.** *There exist positive integers  $k$  and  $n$  such that  $A$  is spanned by the image of  $NS(k, n)$ .*

*Proof.* As before, let  $k$  be the maximal length of monomials in  $g$ ; that is  $k = |g|$ .

By Lemmata 3.1 and 4.2, there exist  $n_1, n_2 \in \mathbb{Z}_{\geq 1}$  and  $\ell_1, \ell_2 \in \mathbb{Z}$  such that for every standard monomial  $m = e_{j_1} \dots e_{j_k}$ , with  $j_1 \leq \dots \leq j_k$ , if  $j_1 \geq n_1$ , then

$$m = \sum c_s m_s + h,$$

where  $h \in I$ , the sum is finite,  $c_s \in \mathbb{K}$ , and the  $m_s$  are standard monomials such that  $i \geq \ell_1$  for each  $e_i$  occurring in  $m_s$ , and  $i < n_1$  for some  $e_i$  occurring in  $m_s$ ; and if  $j_k \leq -n_2$ , then

$$m = \sum c_s m_s + h,$$

where  $h \in I$ , the sum is finite,  $c_s \in \mathbb{K}$ , and the  $m_s$  are standard monomials such that  $i \leq \ell_2$  for each  $e_i$  occurring in  $m_s$ , and  $i > -n_2$  for some  $e_i$  occurring in  $m_s$ .

Let  $n = \max\{n_1, n_2, |\ell_1|, |\ell_2|\}$ . Repeatedly using the observations above to rewrite  $m$  modulo  $I$ , we obtain the result.  $\square$

For the rest of the section, let  $k, n$  be as given by Lemma 4.4, and let  $NS = NS(k, n)$ . We call the elements of  $NS$  *normal words*, and a representation of  $m \in A$  as a linear combination of (images of) normal words a *normal form* for  $m$ , bearing in mind that this normal form may not be unique.

The growth of  $NS$  is polynomial, as we next show.

**Lemma 4.5.** *For any positive integer  $C$ , define a function*

$$\delta_C : \{ \text{standard monomials in } U(W) \} \rightarrow \mathbb{N}$$

by  $\delta_C(e_{i_1} \dots e_{i_k}) = |i_1| + \dots + |i_k| + C$ . For any  $N \in \mathbb{N}$ , let

$$p_C(N) = \#\{w \in NS \mid \delta_C(w) \leq N\}.$$

Then the function  $p_C$  has polynomial growth.

*Proof.* The growth of  $p_C$  does not depend on  $C$ , so without loss of generality let  $C = 1$  and let  $p = p_1$  and  $\delta = \delta_1$ . For a standard monomial  $m$ , we refer to  $\delta(m)$  as the *absolute degree* of  $m$ . Note that  $U(W)$  is *not* graded with respect to absolute degree.

Clearly  $p(N) \leq q(N)r(N)^2$ , where  $q(N)$  is the number of standard monomials in

$$e_{1-n}, \dots, e_0, e_1, \dots, e_{n-1}$$

of absolute degree at most  $N$ , while  $r(N)$  is the number of standard monomials in  $e_n, e_{n+1}, \dots$  of degree at most  $N$  and of length at most  $k - 1$ .

Now, absolute degree, as a function on standard monomials, is always greater than or equal to length. Thus  $q(N)$  does not exceed the number of standard monomials in  $e_{1-n}, \dots, e_{n-1}$  of length at most  $N$ , which is equal to  $\binom{N+2n-1}{N} \leq cN^{2n-1}$ , for some positive constant  $c$  which depends only on  $n$  and not on  $N$ .

On the other hand, in a monomial in  $e_n, e_{n+1}, \dots$  of absolute degree at most  $N$  and of length at most  $k - 1$ , there are no more than  $N$  options for each letter and therefore  $r(N) \leq N^{k-1}$  for all  $N$ . Hence  $p(N) \leq cN^{2n+2k-3}$  and has polynomial growth.  $\square$

**4.2. Choice of filtration.** It remains to estimate the growth of  $A$  from the spanning set constructed in the previous subsection.

Since  $A$  is finitely generated, the growth of any finite filtration bounds the growth of  $A$ . The main result of this subsection is that there is a constant  $C$  such that the function  $\delta_C$  induces a finite filtration of  $A$ , which by Lemma 4.5 will have polynomial growth.

To have a filtration  $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$  on  $A = \bigcup A_i$  means to choose a map  $\rho : A \rightarrow \mathbb{N}$ , satisfying

$$(4.6) \quad \rho(uv) \leq \rho(u) + \rho(v), \rho(u + v) \leq \max\{\rho(u), \rho(v)\} \text{ and } \rho(\alpha u) = \rho(u),$$

for any  $u, v \in A$  and  $\alpha \in \mathbb{K}^*$ . Suppose that for some  $C$ , the map  $\delta_C : NS \rightarrow \mathbb{N}$  has the property that for any two normal words  $m_1, m_2 \in NS$ , we can find a normal form

$$\pi(m_1 m_2) = \sum c_i \pi(w_i),$$

where the  $c_i \in \mathbb{K}$  and the  $w_i$  are normal words so that

$$(4.7) \quad \delta_C(w_i) \leq \delta_C(m_1) + \delta_C(m_2) \text{ for all } i.$$

We claim that this is enough to construct the required  $\rho$ . For, define

$$\rho(u) = \min_{\substack{\text{normal forms} \\ u = \sum c_j \pi(w_j)}} \left\{ \max_j \delta_C(w_j) \right\}.$$

Then  $\rho$  is easily seen to satisfy the required conditions (4.6).

So our goal is to show that there is some constant  $C$  so that  $\delta_C$  satisfies (4.7).

**Proposition 4.8.** *Let  $\ell = \max\{|i| : e_i \text{ features in } g\}$ . Then (4.7) holds for  $C = 4k^2\ell$ , where we recall that  $k = |g|$ .*

*Proof.* It suffices to show that for any normal words  $m_1, m_2 \in NS$ , and for any normal form  $\pi(m_1 m_2) = \sum c_i \pi(w_i)$  for  $\pi(m_1 m_2)$ , where  $w_i \in NS$  and  $c_i \in \mathbb{K}^*$ , that

$$(4.9) \quad \delta_0(w_i) \leq \delta_0(m_1) + \delta_0(m_2) + C \quad \text{for all } i.$$

So we need to understand how  $\delta_0$  behaves on the words appearing in a normal form for  $\pi(m_1 m_2)$ .

Recall the definition of  $n$  from Lemma 4.4. Write

$$m_1 = a_1 u_1 b_1 \text{ and } m_2 = a_2 u_2 b_2,$$

where  $u_1$  and  $u_2$  are standard monomials of any length on variables with indices strictly between  $-n$  and  $n$ ,  $a_1$  and  $a_2$  are standard monomials of length  $< k$  on letters with indices  $\leq -n$ , while  $b_1, b_2$  are standard monomials of length  $< k$  on letters with indices  $\geq n$ . Now, normal words are standard monomials, and we first use the commutation relations  $e_i e_j = e_j e_i + (j - i) e_{i+j}$  to write  $m_1 m_2$  as a linear combination of standard monomials, that is as a linear combination of words of the form

$$m_3 = a_3 u_3 b_3,$$

where  $a_3$  is a standard monomial on letters with indices  $\leq -n$ ,  $u_3$  is a standard monomial on letters with indices strictly between  $-n$  and  $n$ , and  $b_3$  is a standard monomial on letters with indices  $\geq n$ . Let us call those letters with indexes  $|i| \geq n$ , *big letters*. Note that in the course of applying the commutation relations, the total number of big letters does not increase. Thus  $a_3$  and  $b_3$  have length  $\leq 2k - 2$ , while  $u_3$  may have any length.

Now we will use the reduction procedure from Lemma 3.1 to see how  $\delta_0$  changes when we get rid of big letters in  $b_3$ . If the length of  $b_3$  is less than  $k$  we do not have to do anything. Otherwise let  $m$  be the monomial composed of the last  $k$  letters of  $b_3$ . According to Lemma 3.1, to get rid of one of the (big) letters in  $m$ , we find a sequence of derivations  $D = \partial_{a_1} \dots \partial_{a_k}$ , with all the  $a_i \geq 1$ , and some  $c \in \mathbb{K}^*$  such that

$$(4.10) \quad cD(g) = m + \sum c_s m'_s.$$

Let  $m'$  be some  $m'_s$ . Now  $m'$  is a standard monomial which falls into one of three cases:

- I.  $|m'| < |m|$ , and  $m'$  is obtained from some monomial  $\tilde{g}$  in  $g$  by applying  $D$  and then the commutation relations;
- II.  $|m'| = |m|$  and  $m'$  is obtained from some monomial  $\tilde{g}$  in  $g$  (which necessarily has length  $k$ ) by a non-permutational action of  $D$ ;
- III.  $|m'| = |m|$  and  $m'$  is obtained from some monomial  $\tilde{g}$  in  $g$  by a permutational action of  $D$ .

First, we note what happens to the number of big letters in each case. In case I, since  $m'$  is shorter than  $m$ , and all letters in  $m$  are big,  $m'$  contains fewer big letters than  $m$ . In case II, since the action is non-permutational, there is a letter in  $m'$  which was present in  $\tilde{g}$ . But monomials of  $g$  consist of letters which are not big, by definition of  $n$ . Thus the number of big letters in  $m'$  is smaller than in  $m$ . In case III, it may be that the number of big letters in  $m'$  is still equal to  $k$ , which is the number of (big) letters in  $m$ , but certainly there are no more than  $k = |m'|$  big letters in  $m'$ . Further, by our choice of  $n$ , in case III all  $e_i$  occurring in  $m'$  have  $i \geq 1$ .

We now consider how  $\delta_0$  changes throughout this process. In situation III, as  $m' \leq m$  and  $m'$  and  $m$  are made of letters  $\geq e_1$ , we have  $\delta_0(m') = \deg(m') \leq \deg(m) = \delta_0(m)$ . In cases I and II,  $\delta_0(m')$  may be bigger than  $\delta_0(m)$ . Since applying the commutation relations does not increase  $\delta_0$ , we may assume that the monomial  $m'$  is a (possibly non standard) monomial obtained from applying  $D$  to a monomial  $\tilde{g}$  of  $g$ .

Recall that  $\bar{g}$  is the  $\prec$ -leading monomial of  $g$ , and  $m$  is obtained from the monomial  $\bar{g}$  by applying  $D$ . Set  $\deg(D) = a_1 + \dots + a_k$ , so applying  $D$  increases the degrees of homogenous

elements by  $\deg(D)$ . As  $\delta_0(e_{a+b}) \leq \delta_0(e_a) + \delta_0(e_b)$  and all the  $a_i$  are  $\geq 1$ , we have

$$\delta_0(m') \leq \delta_0(\tilde{g}) + \deg D.$$

So

$$\begin{aligned} \delta_0(m') - \delta_0(m) &\leq \delta_0(\tilde{g}) + \deg D - \delta_0(m) \\ &= \delta_0(\tilde{g}) + \deg D - \deg m \quad \text{as letters in } m \text{ are big} \\ &\leq \delta_0(\tilde{g}) + |\deg m - \deg D| \\ &= \delta_0(\tilde{g}) + |\deg \bar{g}| \\ &\leq 2k\ell, \end{aligned}$$

by choice of  $\ell$ , as  $\tilde{g}$  and  $\bar{g}$  have no more than  $k$  letters.

To summarize the discussion above: in cases I and II, we remove at least one big letter and increase  $\delta_0$  by no more than  $2k\ell$ . In case III, we do not remove big letters and do not increase  $\delta_0$ . As proved in Lemma 3.1, after repeating this procedure finitely many times, we may write  $m_1m_2$ , modulo  $I$ , as a linear combination of normal words. However many times we repeat the procedure we remove a maximum of  $k-1$  big letters from  $b_3$ , and thus we add a maximum of  $2k(k-1)\ell$  to  $\delta_0$ . Note that in applying this process to  $b_3$ , we never add any letters with indices  $< -n$ .

After we apply the procedure from Lemma 4.2 to  $a_3$  at the other end of the word, we have found a normal form for  $m_1m_2$  and have added maximum of another  $2k(k-1)\ell$  to  $\delta_0$ . In other words, we have written

$$\pi(m_1m_2) = \sum c_i \pi(w_i)$$

where the  $w_i$  satisfy (4.9), as required.  $\square$

The proof of Theorem 4.1 is now an easy combination of other results in this section.

*Proof of Theorem 4.1.* Let  $C$  be the constant given by Proposition 4.8. The discussion before that proposition shows that setting

$$A(N) = \text{span}\{\pi(w) : w \in NS, \pi_C(w) \leq N\}$$

defines a finite filtration on  $A$ . By Lemma 4.5, this filtration has polynomial growth, and so  $\text{GKdim}(A) < \infty$ .  $\square$

Theorem 4.1 gives the second part of Theorem 1.2, dealing with  $U(W)$ .

**Remark 4.11.** Let  $\mathbf{W}_1$  be the first *Cartan algebra*, which is isomorphic to the subalgebra of  $W$  spanned by  $\{e_n : n \geq -1\}$ . This is a simple graded Lie algebra of polynomial growth. Similar methods to those used in Theorem 4.1 show that  $U(\mathbf{W}_1)$  has just infinite GK-dimension. To show this for all simple graded Lie algebras of polynomial growth is the subject of ongoing work.

## 5. CENTRAL QUOTIENTS OF THE ENVELOPING AND SYMMETRIC ALGEBRAS OF $Vir$

In this section we first prove that all the central quotients  $U(Vir)/(c-\kappa)$  have just infinite GK-dimension, completing the proof of Theorem 1.2, and then consider the related Poisson algebras  $S(Vir)/(c-\kappa)$ . Because the ideas of the proofs are similar to those in previous sections, we leave some details to the reader.

**5.1. Central quotients of the enveloping algebra of  $Vir$ .** Fix  $\kappa \in \mathbb{K}$  and let  $R = U(Vir)/(c - \kappa)$ . Essentially the same argument as for the full Witt algebra works to show that  $R$  has just infinite GK-dimension; we give a sketch of the proof.

Note that  $R$ , like  $U(W)$ , has a basis of standard monomials in the  $e_i$ .

**Definition 5.1.** For positive integers  $k, n$ , let  $NS(k, n) \subset R$  be the set of standard monomials  $m$  in the  $e_i$  which admit a factorisation  $m = aub$ , where  $a$  is a standard monomial of length  $< k$  in  $e_{-n}$  and smaller letters,  $u$  is a standard monomial in  $e_{1-n}, \dots, e_0, e_1, e_2, \dots, e_{n-1}$ , and  $b$  is a standard monomial of length  $< k$  in  $e_n$  and bigger letters.

**Lemma 5.2.** *Let  $g$  be a nonzero element of  $R$ , let  $I = (g)$ , and let  $A = R/I$ . There exist positive integers  $k$  and  $n$  such that  $A$  is spanned by the image of  $NS(k, n)$ .*

*Proof.* The key point is that the reduction formulae in Lemmata 3.1 and 4.2 still hold. As before, let  $\partial_a = [\_, e_a]$  as a linear operator on  $R$  and consider the effect of applying some  $\partial_{a_1} \cdots \partial_{a_k}$  to a standard monomial of length  $k$ . If  $\kappa \neq 0$  and some expression of the form  $[e_{-a_j}, e_{a_j}]$  has been computed, we may obtain some standard monomials of length  $< k$  in the result; but the leading term will have length  $k$  and will be given by the procedures in the previous section. Thus the proof of Lemma 4.4 goes through in this situation, almost without change.  $\square$

**Theorem 5.3.** *Let  $I = (g)$  be a two-sided ideal in  $R$  generated by one nonzero element  $g \in R$ . Then  $A = R/I$  has polynomial growth.*

*Proof.* We may define the functions  $\delta_C$  just as with  $U(W)$ . The only part of the argument which is different for  $R$  is the proof of Proposition 4.8. When we compute

$$(5.4) \quad c\partial_{a_1} \cdots \partial_{a_k}(g) = m + \sum_s c_s m'_s,$$

as in (4.10), consider some  $m' = m'_s$  as before. In addition to cases I, II, III as in the proof of Proposition 4.8, we may have

I'.  $|m'| < |m|$ , and  $m'$  is obtained from some monomial  $\tilde{g}$  in  $g$  by applying  $\partial_{a_1} \cdots \partial_{a_k}$  and then applying the relation

$$[e_{-i}, e_i] = 2ie_0 + \frac{i - i^3}{12} \delta_{i+j,0} \kappa.$$

As before, applying this relation does not increase  $\delta_0$ , so we may assume that  $m'$  is a (possibly non standard) monomial obtained from applying  $D$  to a monomial  $\tilde{g}$  of  $g$ . The argument of Proposition 4.8 goes through with only minor changes, and the result follows just as in the proof of Theorem 4.1.  $\square$

Theorem 5.3 completes the proof of Theorem 1.2 by establishing the part dealing with  $U(Vir)$ .

**Remark 5.5.** By the same method as in the proof of Theorem 5.3, one may show that the localised enveloping algebra  $U(Vir) \otimes_{\mathbb{K}[c]} \mathbb{K}(c)$ , considered as an algebra over  $\mathbb{K}(c)$ , has just infinite GK-dimension. It follows, using a similar argument to the proof of [KL00, Lemma 3.10], that  $U(Vir) \otimes_{\mathbb{K}[c]} \mathbb{K}(c)$  has just infinite GK-dimension considered as a  $\mathbb{K}$ -algebra. We leave the details to the reader.

**5.2. Central quotients of the symmetric algebra of  $Vir$ .** In this subsection, let  $\kappa \in \mathbb{K}$  and let  $R = S(W)/(c - \kappa)$ . Since  $c - \kappa$  is Poisson central,  $R$  is a Poisson algebra; in fact if we filter  $U(Vir)$  by setting  $|e_i| = 1$  and  $|c| = 0$ , then  $R$  is the associated graded ring of  $U(Vir)/(c - \kappa)$ . Note that if we define  $d_a = \{ \_, x_a \}$  and  $\partial_a = [ \_, e_a ]$  as before, then (3.3) still holds.

Similar arguments to those that have gone before prove:

**Theorem 5.6.** *Let  $g$  be a nonzero element of  $R$  and let  $I = \{(g)\}$  be the Poisson ideal generated by  $g$ . Then  $A = R/I$  has polynomial growth.*

*Proof.* The reduction process works as before: writing  $g = \text{gr}(g')$  and computing

$$h = c\partial_{a_1} \dots \partial_{a_k}(g') = m + \sum_s c_s m'_s$$

as in (5.4), by (3.3)

$$\text{gr}(h) = \text{gr}(m) + \sum_t c_t \text{gr}(m'_t),$$

where the only  $m'_t$  surviving have length  $k$ . Thus as in Lemma 5.2 there are  $k$  and  $n$  so that  $A$  is spanned by the image of  $\text{gr}(NS(k, n))$ .

Comparing  $\delta_0(\text{gr}(m'))$  with  $\delta_0(\text{gr}(m))$  as in the proof of Proposition 4.8 we see that only cases II and III occur. The conclusion of Proposition 4.8 still holds, and so as in the proof of Theorem 4.1  $\text{GKdim}(A) < \infty$ .  $\square$

**Remark 5.7.** Similarly, one may show that  $S(\mathbf{W}_1)$  has just infinite GK-dimension. We omit the proof.

## 6. APPLICATIONS

In this section we give several applications of Theorem 1.2. We first give a short proof that Verma modules for  $Vir$  are faithful over the appropriate central factor of  $U(Vir)$ . (A more direct proof is an unpublished result of Nolan Wallach [WS13].) We next prove that  $U(W_+)$ ,  $U(W)$ , and  $U(Vir)$  all satisfy the ascending chain condition on completely prime ideals. As a consequence, these algebras are *Hopfian*: they are not isomorphic to any proper quotient.

**6.1. Annihilators of induced modules.** Fix  $\lambda, \kappa \in \mathbb{K}$ . Note that the Virasoro algebra  $Vir$  has a triangular decomposition: define  $\mathfrak{n}_+ := \mathbb{K}(e_n : n \geq 1)$ ,  $\mathfrak{h} := \mathbb{K}(c, e_0)$ , and  $\mathfrak{n}_- := \mathbb{K}(e_n : n \leq -1)$ . Let  $\mathfrak{b}_+ := \mathfrak{n}_+ \oplus \mathfrak{h}$ . Let  $\mathbb{K}_{\kappa, \lambda}$  be the one-dimensional representation of  $\mathfrak{b}_+$  where  $\mathfrak{n}_+$  acts trivially,  $c$  acts as  $\kappa$ , and  $e_0$  acts as  $\lambda$ . Then define the *Verma module*  $M_{\kappa, \lambda}$  to be  $U(Vir) \otimes_{U(\mathfrak{b}_+)} \mathbb{K}_{\kappa, \lambda}$ . It is immediate that

$$M_{\kappa, \lambda} \cong U(Vir)/U(Vir)(c - \kappa, e_0 - \lambda, e_n : n \geq 1)$$

and that  $M_{\kappa, \lambda}$  is non-positively graded, with  $\dim(M_{\kappa, \lambda})_{-n} = \mathcal{P}(n)$ , the  $n$ 'th partition number.

Verma modules are examples of the larger class we call, slightly imprecisely, *induced modules*. These are modules of the form  $M = U(Vir) \otimes_{U(\mathfrak{b}_+)} M'$ , where  $M'$  is a representation of  $\mathfrak{b}_+$ . Besides Verma modules, examples include *logarithmic representations*, where  $\dim_{\mathbb{K}} M' < \infty$  and where  $\mathfrak{n}_+$  acts trivially on  $M'$ ,  $c$  acts as a scalar, and  $e_0$  acts as a non-semisimple matrix. These representations are important in logarithmic conformal field theory, see [GK96].

*Whittaker modules* [OW09] form another class of examples. Here let  $\mathfrak{n}' = \mathfrak{n} \oplus \mathbb{K}c$  and let  $M''$  be the one-dimensional  $\mathfrak{n}'$  module where  $c$  acts as a scalar, and the  $e_n$  act trivially for  $n \geq 3$ . The module

$$M = U(\text{Vir}) \otimes_{U(\mathfrak{n}')} M'' \cong U(\text{Vir}) \otimes_{U(\mathfrak{b}_+)} U(\mathfrak{b}_+) \otimes_{U(\mathfrak{n}')} M''$$

is a *Whittaker module*. If  $e_1, e_2$  act nontrivially on  $M''$ , then  $M$  is simple by [OW09, Corollary 4.5]. All of these examples are annihilated by some  $c - \kappa$ , where  $\kappa \in \mathbb{K}$ , and so have *central character*  $\kappa$ .

Using Theorem 1.2, we may immediately compute the annihilator of an induced module.

**Theorem 6.1.** *Let  $M'$  be a representation of  $\mathfrak{b}_+$  with central character  $\kappa \in \mathbb{K}$ . Let  $M = U(\text{Vir}) \otimes_{U(\mathfrak{b}_+)} M'$ . Then  $\text{Ann}_{U(\text{Vir})} M = (c - \kappa)$ .*

*Proof.* Clearly  $(c - \kappa)M = 0$ .

Let  $\mathcal{P}$  be the set of *negative partitions*  $\underline{\lambda} = (\lambda_1, \dots, \lambda_k)$  where the  $\lambda_i$  are negative integers with  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k$ . If  $\underline{\lambda} = (\lambda_1, \dots, \lambda_k) \in \mathcal{P}$ , let  $e_{\underline{\lambda}} = e_{\lambda_1} \dots e_{\lambda_k}$ . If  $0 \neq m \in M'$ , it follows from the Poincaré-Birkhoff-Witt theorem that the elements  $\{e_{\underline{\lambda}} \otimes m : \underline{\lambda} \in \mathcal{P}\}$  are linearly independent in  $M$ , and thus, as these elements are in bijection with partitions, that  $M$  has subexponential growth and infinite Gelfand-Kirillov dimension.

Let  $K = \text{Ann}_{U(\text{Vir})} M$ . If  $K \not\supseteq (c - \kappa)$  then by Theorem 1.2  $U(\text{Vir})/K$  has polynomial growth and thus, by [KL00, Proposition 5.1(d)], so does  $M$ . This contradiction shows that  $K = (c - \kappa)$ .  $\square$

**Corollary 6.2.** *For any  $\kappa, \lambda \in \mathbb{K}$ , the Verma module  $M_{\kappa, \lambda}$  is a faithful  $U(\text{Vir})/(c - \kappa)$ -module.*  $\square$

Corollary 6.2 is an unpublished result of Nolan Wallach [WS13], and is independently due to Olivier Mathieu in unpublished work; see [CM07, footnote 2, p. 496]. We thank Rupert Wei Tze Yu for pointing out this reference to us.

It is known [FF84, Theorem 1.2] that for any  $\kappa$ , the module  $M_{\kappa, \lambda}$  is simple for generic  $\lambda$ . Thus it follows immediately that  $U(\text{Vir})/(c - \kappa)$  is primitive.

**Corollary 6.3.** *Let  $N$  be a logarithmic representation or a Whittaker module over  $\text{Vir}$ . Then  $\text{Ann}_{U(\text{Vir})}(N) = (c - \kappa)$  for some  $\kappa \in \mathbb{K}$ .*  $\square$

**6.2. Completely prime ideals.** In [PS17, Conjecture 1.3], it is conjectured that  $U(W_+)$  satisfies the ascending chain condition on two-sided ideals. We cannot prove this, but we do show

**Proposition 6.4.** *The algebras  $U(W_+)$  and  $U(\text{Vir})$  satisfy the ascending chain condition (ACC) on completely prime ideals.*

*Proof.* We first note that any ring  $R$  of finite or just infinite GK-dimension satisfies ACC on completely prime ideals. Letting  $P_0$  be the first ideal in the chain, it is sufficient to show that  $R/P_0$  has ACC on completely prime ideals. Thus we may replace  $R$  by  $R/P_0$  and assume that  $R$  is a domain with  $\text{GKdim } R < \infty$ . Now if  $I$  is a nonzero ideal of  $R$ , then by [KL00, Proposition 3.15],  $\text{GKdim } R/I \leq \text{GKdim } R - 1$ , so by induction the length of a chain of completely prime ideals is bounded by  $\text{GKdim } R$ . Thus by Theorem 1.2,  $U(W_+)$  and  $U(\text{Vir})/(c - \kappa)$  (for any  $\kappa \in \mathbb{K}$ ) have ACC on completely prime ideals.

In fact, note that if  $f[x] \in \mathbb{K}[x]$  is irreducible, then  $U(\text{Vir})/(f(c))$  has just infinite GK-dimension and thus ACC on completely prime ideals. To see this, let  $\mathbb{K}'$  be the extension



field  $\mathbb{K}[x]/f(x)$  of  $\mathbb{K}$ , and note that

$$U(\text{Vir})/(f(c)) = U_{\mathbb{K}}(\text{Vir})/(f(c)) \cong U_{\mathbb{K}'}(\text{Vir})/(c - x).$$

This last has just infinite GK-dimension by Theorem 1.2.

We now consider an ascending chain  $P_1 \subseteq P_2 \subseteq \cdots$  of completely prime ideals of  $U(\text{Vir})$ . If  $\bigcup P_n$  contains a nonzero element of  $\mathbb{K}[c]$ , then as the  $P_n$  are prime and  $c$  is central, some  $P_n$  contains an irreducible polynomial  $f(c) \in \mathbb{K}[c]$ . By the first part of the proof, therefore, the chain stabilizes.

So we may assume that each  $P_n \cap \mathbb{K}[c] = 0$ . As  $P_n$  is prime, each  $U(\text{Vir})/P_n$  is  $\mathbb{K}[c]$ -torsionfree. Thus if  $P_n \neq P_{n+1}$ , then  $(P_{n+1}/P_n) \otimes_{\mathbb{K}[c]} \mathbb{K}(c) \neq 0$  and so

$$P_n \otimes_{\mathbb{K}[c]} \mathbb{K}(c) \neq P_{n+1} \otimes_{\mathbb{K}[c]} \mathbb{K}(c).$$

Further, these ideals are completely prime as

$$U(\text{Vir}) \otimes_{\mathbb{K}[c]} \mathbb{K}(c)/P_n \otimes_{\mathbb{K}[c]} \mathbb{K}(c) \cong (U(\text{Vir})/P_n) \otimes_{\mathbb{K}[c]} \mathbb{K}(c)$$

is a domain.

Thus it suffices to show that  $U(\text{Vir}) \otimes_{\mathbb{K}[c]} \mathbb{K}(c)$  has ACC on completely prime ideals. By Remark 5.5,  $U(\text{Vir}) \otimes_{\mathbb{K}[c]} \mathbb{K}(c)$  has just infinite GK-dimension. Thus by the first part of the proof,  $U(\text{Vir}) \otimes_{\mathbb{K}[c]} \mathbb{K}(c)$  satisfies the ACC on completely prime ideals.  $\square$

**6.3. The Hopfian and Bassian properties.** To end the paper, we consider two ring-theoretic properties which are related to noetherianity. A ring  $R$  is *Hopfian* if  $R$  is not isomorphic to any proper quotient  $R/J$  (equivalently, any epimorphism from  $R \rightarrow R$  is an isomorphism). More strongly,  $R$  is *Bassian* if there is no injection of  $R$  into any proper quotient  $R/J$ . We thank Lance Small for introducing us to these concepts.

**Proposition 6.5.** *The algebras  $U(W_+)$ ,  $U(W)$ ,  $U(\mathbf{W}_1)$ , and  $U(\text{Vir})/(c - \kappa)$  are Bassian and Hopfian, and  $U(\text{Vir})$  is Hopfian.*

That  $U(W_+)$  is Hopfian is proved in [RS17, Remarks 2.2], and [RS17, Section 4] asks whether  $U(W)$  is Bassian or Hopfian.

*Proof.* If  $R$  has just infinite GK-dimension, then  $\text{GKdim } R/J < \text{GKdim } R$  for any proper ideal  $J$  of  $R$ , so  $R$  cannot inject in  $R/J$ . Thus the Bassian (and thus Hopfian) property for  $U(W_+)$ ,  $U(W)$ , and  $U(\text{Vir})/(c - \kappa)$  follows from Theorem 1.2. For  $U(\mathbf{W}_1)$  it follows from Remark 4.11.

To show that  $U(\text{Vir})$  is Hopfian, let  $R = U(\text{Vir})$  and let  $f$  be a surjective endomorphism of  $R$ , with kernel  $J$ . As  $R/J \cong \text{Im}(f)$  is torsionfree as a module over  $\mathbb{K}[c]$ , the complex

$$0 \rightarrow J \otimes_{\mathbb{K}[c]} \mathbb{K}(c) \rightarrow R \otimes_{\mathbb{K}[c]} \mathbb{K}(c) \rightarrow (R/J) \otimes_{\mathbb{K}[c]} \mathbb{K}(c) \rightarrow 0$$

is exact. Now by Remark 5.5, we must have  $J \otimes_{\mathbb{K}[c]} \mathbb{K}(c) = 0$ . As  $R$  is  $\mathbb{K}[c]$ -torsionfree,  $J = 0$ .  $\square$

That  $U(\text{Vir})$  is Hopfian also follows from [RS17, Corollary 2.6] and Proposition 6.4. We do not know whether  $U(\text{Vir})$  is Bassian.

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